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Existence of Best Mean Rational Approximations

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Conditions are given which guarantee the existence of a best approximation by generalized rational functions with respect to a generalized integral norm, which includes as a special case all L_p norms.

Let τ be a nonnegative continuous function. Let $\int$ denote the integral on $[a, b]$. For measurable g on $[a, b]$ define $\| g \| = \int \tau(g)$.

Let P and Q be two finite-dimensional linear subspaces of $C[a, b]$. Define $R = \{p/q\colon p \in P,\ q \in Q,\ q \not\equiv 0\}$. The approximation problem is given a bounded measurable f, to find an element $r^* \in R$ minimizing $\| f - r \|$ over all $r \in R$. Such an element r^* is called a best approximation.

1. BACKGROUND

"Norms" of this type with a less general τ were considered by Walsh and Motzkin [4] for linear approximation. A case of special interest arises when

$$\tau(t) = | t |^p \qquad 0 < p < \infty.$$

In this case, the approximation problem is a problem of best L_p approximation.

If g is measurable, then $\tau(g)$ is measurable and $\| g \|$ is well defined; it may, however, be $+\infty$. There is no reason to require that g be integrable, since this does not ensure that $\tau(g)$ is integrable.

The family of rational functions which is most often considered is the family $\{p/q : p \in P, q \in Q, q > 0\}$ of "admissible" rational functions. This family has many desirable properties, including continuity of all its elements. However, in order to guarantee existence of best approximations we must consider a larger family, such as R or

$$\hat{R} = \{p/q : p \in P, q \in Q, q \geqslant 0, q \not\equiv 0\}.$$

An examination of the existence proofs of this note shows that they hold also for $\hat{R}$ (providing, of course, it is nonempty).

Often R (or $\hat{R}$) will contain elements which are not continuous. There is, therefore, little to be gained in requiring that f be continuous. To ensure that $f - r$ is measurable, we must verify that r is, which we do later.

One difficulty with generalized rational functions is the zeros of their denominators. One way of getting around this difficulty is to adopt a hypothesis similar to that of Boehm [1] for Chebyshev approximation. We shall say that Q has the *zero-measure property* if for any $q \in Q$, $q \not\equiv 0$, the set

$$Z(q) = \{x : q(x) = 0\}$$

is of measure zero. This hypothesis is satisfied by the common linear families used for denominators, such as polynomials, sums of exponentials, and trigonometric polynomials. This hypothesis makes the values of p/q at the zeros of q completely irrelevant to $\int \tau(f - r)$ and so we can assign any value to r at the zeros of its denominator.

The results of this paper are generalizations of the existence, theorems in Refs. [4, pp. 1228–1229] and [5, pp. 357–358]. In the first of these, an existence theorem is given for linear approximation with respect to a restricted class of τ-norms. In the second an existence theorem is proved for L_p approximation by (ordinary) rational functions. Existence theories for Chebyshev approximation by generalized rational functions can be found in Refs. [1, 2, 6, 7].

2. Measurability of Rational Functions

Lemma 1. *Let p, q be continuous on $[a, b]$ and let $Z(q)$ be of measure zero. Then p/q is measurable on $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. $Z(q)$ is a closed set and so $[a, b] \sim Z(q)$ is open. It is a countable union of disjoint open intervals. Select a finite set U of these intervals such that $\text{meas}(U) > \text{meas}[a, b] - \epsilon/2$. Select a closed subset F of U such that $\text{meas}(F) > \text{meas}[a, b] - \epsilon$. Set $g(x) = p(x)/q(x)$ for $x \in F$. Then g is continuous on F and has a continuous extension to $[a, b]$. By Luzin's Theorem [3, p. 41], p/q is measurable.

3. Existence on an Interval

It is useful to parametrize R. Let $\{p_1, ..., p_n\}$, $\{q_1, ..., q_m\}$ be bases for P and Q, respectively, and define

$$R(A, x) := P(A, x)/Q(A, x) := \sum_{k=1}^{n} a_k p_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} q_k(x),$$

$$\| A \| = \max\{| a_i | : i \leqslant i \leqslant n\}.$$

Without loss of generality we can normalize $R(A, x)$ so that

$$\sum_{k=1}^{m} | a_{k+n} | = 1. \tag{2}$$

LEMMA 2. *If* $\{\| A^k \|\} \to \infty$, *then there exists a nondegenerate closed interval I such that*

$$M_k = \inf\{| f(x) - R(A^k, x)| : x \in I\} \to \infty \qquad as \qquad k \to \infty.$$

Proof. By the argument given at the start of the proof of Theorem 1 of Ref. [2], $\|A^k\| \to \infty$ implies $\inf\{ | P(A^k, x) | : x \in I\} \to \infty$ for some nondegenerate closed interval I. Now

$$\inf\{| R(A_k , x)| : x \in I\} \leqslant \inf\{| P(A^k, x)| : x \in I\} \Big/ \sum_{j=1}^{m} \{\sup | q_j(y)| : y \in I\}$$

and the right side tends to infinity as $k \to \infty$. The conclusion of the lemma follows easily.

THEOREM 1. *Let Q have the zero measure property and let* $\tau(t) \to \infty$ *as* $| t | \to \infty$. *Then there exists a best approximation to each bounded measurable function f.*

Proof. Let

$$\rho(f) = \inf \left(\| f - R(A, \cdot)\| : \sum_{k=1}^{m} | a_{n+k} | = 1 \right).$$

If this infimum is infinite any approximation is best. Otherwise, let $\{\|f - R(A^k, \cdot)\|\}$ be a decreasing sequence with limit $\rho(f)$. If $\{\| A^k \|\}$ is unbounded we can assume that $\{\| A^k \|\} \to \infty$ and, using Lemma 2, we have

$$\int \tau(f - R(A^k, \cdot) \geqslant \int_I \tau(f - R(A^k, \cdot)) \geqslant \int_I \min\{\tau(f(x) - R(A^k, x)) : x \in I\},$$

where I is as in Lemma 2. The extreme right side tends to infinity as $k \to \infty$. It follows that $\| f - R(A^k, \cdot)\| \to \infty$, a contradiction. Hence $\{\| A^k \|\}$ is bounded and $\{A^k\}$ has a limit point A; we may assume $\{A^k\} \to A$. Then $\{R(A^k, \cdot)\}$ converges, except on $Z(Q(A, \cdot))$, to $R(A, \cdot)$. Hence $\tau(f - R(A^k, \cdot))$ converges pointwise to $\tau(f - R(A, \cdot))$ except on $Z(Q(A, \cdot))$, a set of measure zero. By Fatou's Theorem [3, p. 59],

$$\int \tau(f - R(A, \cdot)) \leqslant \lim_{k\to\infty} \int \tau(f - R(A_k , \cdot)) = \rho(f).$$

4. Approximation by Admissible Rational Functions

In some cases we can show that a best approximation exists which is admissible. We modify a definition of Cheney and Goldstein [6, p. 234].

Definition. The triple P, Q, $\| \quad \|$ is said to have property (C) if $p \in P$, $q \in Q$, f measurable and bounded on $[a, b]$, $\| f - p/q \| = \lambda$ imply that there exist $p_0 \in P$, $q_0 \in Q$ with $q_0 > 0$ such that

$$\| f - P_0/q_0 \| \leqslant \lambda. \tag{3}$$

An immediate consequence of Theorem 1 is

Corollary. *Let P, Q, $\| \quad \|$ have property* (C) *and let Q have the zero measure property. Then an admissible best approximation exists to each bounded measurable function f.*

We now apply the corollary to the most common case of interest.

Theorem 2. *Let there exist an $\alpha > 0$ such that $\tau(t) \geqslant \alpha \mid t \mid$. Let f be a bounded measurable function and let P, Q be, respectively, the families of polynomials of degree $n - 1$, $m - 1$. There exists an admissible best approximation to f.*

Proof. We verify property (C). Suppose $r = p/q$ has a pole. Define $\| \quad \|_1$ to be the L_1 norm. Now

$$\| f - r \| \geqslant \alpha \| f - r \|_1 \geqslant \alpha \| r \|_1 - \alpha \| f \|_1 ,$$

and as $\| r \|_1 = \infty$, $\| f - r \| = \infty$. Hence (3) is always satisfied. If p/q has no poles there exists an admissable rational function p_0/q_0 differing from it on only finitely many points and so $\| f - p_0/q_0 \| = \| f - p/q \|$. Thus property (C) holds; the zero measure property obviously holds for Q.

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